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Notes

A Simple Heuristic Method for Analyzing the Effect of Boundary Conditions on Numerical Stability*

It is shown that, contrary to the prevalent view, the Fourier method can be used to obtain information about the effect of boundary conditions on numerical stability.

Numerical solutions to problems in such fields as fluid mechanics and heat transfer are commonly generated by means of finite-difference equations that approximate the governing partial differential equations and boundary conditions. Investigations of the numerical stability of such finite-difference equations, including the effects of the boundary conditions, are of paramount importance.

The most widely used method for performing numerical stability analyses is the Fourier (or Von Neumann) method [1–4], which is simple and convenient to use. This method has been shown to yield necessary (and in certain cases sufficient) conditions for stability of pure linear initial-value problems with constant coefficients [2]. However, such problems form a very small subset of the problems to which the method is actually applied in practice. Practical problems typically involve variable coefficients, nonlinearities, and various types of boundary conditions. When applied to such problems, the Fourier method becomes heuristic rather than rigorous, but is still found to yield much useful information. This usefulness is not merely fortuitous, but is to be expected on the following basis [1]: If the complete difference equations are linearized about a small neighborhood in space and time, then the conditions for the applicability of the Fourier method are approximately satisfied *locally* (at least for short wavelengths and high frequencies), even though they are not satisfied globally for the problem as a whole. Thus, it is intuitively reasonable to expect a linearized local application of the Fourier method to yield essentially correct necessary stability conditions, even in complex problems where the method does not rigorously apply. This intuitive expectation has been borne out by a wealth of numerical evidence. Since rigorous methods of stability analysis are not available for most problems of practical interest, the use of the Fourier method in this heuristic local manner has become a well-established practice within the computing community.

The statement is frequently made [2–4] that the Fourier method provides no

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information about the influence of boundary conditions on numerical stability, and that to obtain such information the less convenient matrix method (or other inconvenient methods, such as the energy method or the Godunov–Ryabenkii criterion) must be used instead. This statement is correct with regard to the rigorous global application of the Fourier method, but it in no way constrains the heuristic local use of the method. Indeed, since the Fourier method provides useful stability information when applied locally at interior points, it is natural to expect a similar local application of the method at the boundaries to yield useful boundary stability information. The purpose of this paper is to call attention to this natural extension of the Fourier method, and to show by example that it is in fact capable of determining correct boundary stability restrictions.

The concept of a *local* Fourier stability analysis is crucial here and requires clarification. A local stability analysis at a given mesh point is performed by assuming a Fourier mode dependence for the relation between the values of the dependent variables at the given mesh point and their values at the neighboring mesh points to which the given point is directly coupled by the difference scheme. Usually, this coupling extends only to the immediately adjacent mesh points. Although the fact is seldom stated explicitly, it is obvious that *for a heuristic local analysis to be relevant for the problem as a whole, it must be performed at every mesh point in the region of computation*. The most restrictive stability condition resulting from these analyses is the one which must be observed for overall stability of the calculation. Ordinarily, all the interior mesh points are equivalent (i.e., they are coupled to their neighbors by equations of identical form), so that only one representative interior mesh point need be considered. (This fact is the reason why one does not usually think in terms of performing a separate local analysis at each mesh point.) However, the boundary points are invariably anomalous in that the structure of the boundary difference equations differs from that of the interior difference equations. Consequently, the boundary points lead to different local stability conditions than do the interior points. In many cases, these boundary stability conditions are more restrictive than the interior stability condition and hence, govern the overall stability behavior.

The above considerations constitute our rationale for applying the Fourier method locally at the boundaries. We now proceed to illustrate the general method by working out a particular simple example. The heat flow equation with unit diffusivity is considered:

$$\partial u/\partial t = \partial^2 u/\partial x^2 \qquad (0 < x < 1), \tag{1}$$

subject to the boundary conditions

$$\partial u/\partial x = h_1(u - v_1), \qquad \text{at} \quad x = 0, \tag{2}$$

$$\partial u/\partial x = -h_2(u - v_2), \qquad \text{at} \quad x = 1, \tag{3}$$

where h_1, h_2, v_1, and v_2 are constants. We shall examine the stability of the following explicit finite-difference approximation to Eqs. (1)–(3):

$$\frac{1}{\Delta t}(u_j^{n+1} - u_j^{n}) = \frac{1}{(\Delta x)^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \qquad (j = 0, 1,..., N), \tag{4}$$

$$(1/2\Delta x)(u_1^n - u_{-1}^n) = h_1(u_0^n - v_1), \tag{5}$$

$$(1/2\Delta x)(u_{N+1}^n - u_{N-1}^n) = -h_2(u_N^n - v_2), \tag{6}$$

where $N\Delta x = 1$, and u_j^n denotes the difference approximation to $u(j\,\Delta x, n\,\Delta t)$. Equations (5) and (6) effectively define the quantities u_{-1}^n and u_{N+1}^n required by Eq. (4) when $j = 0$ and $j = N$. Because of Eqs. (5) and (6), the structure of Eq. (4) for $j = 0$ and $j = N$ differs from that for $j = 1, 2,..., N - 1$. This difference may be seen explicitly by rewriting Eqs. (4)–(6) in the form

$$\frac{1}{\Delta t}(u_j^{n+1} - u_j^{n}) = \frac{1}{(\Delta x)^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \qquad (j = 1, 2,..., N-1), \tag{7}$$

$$\frac{1}{\Delta t}(u_j^{n+1} - u_j^{n}) = \frac{2}{(\Delta x)^2}(u_{j+1}^n - a_1 u_j^n) + \frac{2h_1 v_1}{\Delta x} \qquad (j = 0), \tag{8}$$

$$\frac{1}{\Delta t}(u_j^{n+1} - u_j^{n}) = \frac{2}{(\Delta x)^2}(u_{j-1}^n - a_2 u_j^n) + \frac{2h_2 v_2}{\Delta x} \qquad (j = N), \tag{9}$$

where $a_1 = 1 + h_1\,\Delta x$ and $a_2 = 1 + h_2\,\Delta x$.

Since Eqs. (7)–(9) differ in their basic structure, their stability properties also may be expected to differ. Each of these three equations, therefore, requires a separate local Fourier stability analysis. The correct stability condition will be the most restrictive of the three conditions resulting from these analyses.

The terms involving v_1 and v_2 in Eqs. (8) and (9) are irrelevant to a linear stability analysis and are therefore omitted in what follows.

The stability analyses of Eqs. (7)–(9) are performed by replacing u_j^n by $\hat{u}\exp(ikj\,\Delta x)\,\xi^n$, where $\hat{u}$ is a constant amplitude factor, k is the wavenumber, and ξ is the growth factor. The stability condition is that $|\,\xi\,| \leqslant 1$ for all k in the interval $[0, \pi/\Delta x]$. Applying this procedure to Eq. (7), we find

$$\xi = 1 - 4(\Delta t/(\Delta x)^2)\sin^2(k\Delta x/2). \tag{10}$$

Similarly, Eqs. (8) and (9) yield

$$\xi = 1 + 2(\Delta t/(\Delta x)^2)(e^{ik\Delta x} - a_1) \tag{11}$$

and

$$\xi = 1 + 2(\Delta t/(\Delta x)^2)(e^{-ik\Delta x} - a_2), \tag{12}$$

respectively. The condition that $|\xi| \leqslant 1$ for all relevant k then leads to the stability restrictions

$$\Delta t/(\Delta x)^2 \leqslant \tfrac{1}{2}, \tag{13}$$

$$\Delta t/(\Delta x)^2 \leqslant 1/(1 + a_1) = 1/(2 + h_1 \Delta x), \tag{14}$$

$$\Delta t/(\Delta x)^2 \leqslant 1/(1 + a_2) = 1/(2 + h_2 \Delta x), \tag{15}$$

corresponding to Eqs. (7)–(9), respectively. Equation (13) is the usual Fourier stability result obtained by disregarding the boundary conditions; that is, by analyzing Eq. (7) alone. The correct stability condition, including the effects of the boundary conditions, is the most restrictive of Eqs. (13)–(15). The condition in Eq. (13) is always the least restrictive and hence, may be ignored; thus, the final stability condition is

$$\Delta t/(\Delta x)^2 \leqslant 1/(2 + h_0 \Delta x), \tag{16}$$

where h_0 is the larger of h_1 and h_2. The preceding result is in agreement with the necessary condition obtained by the matrix method [3] and can be shown to be a sufficient condition by applying the energy method to the difference Eqs. (4)–(6).

The preceding example illustrates the general application of the local Fourier method to the problem of determining the effect of boundary conditions on numerical stability. The basic idea is very simple: The Fourier method, in addition to being applied locally to the difference equations used in the interior of the computation region, is also applied to the difference equations used on the boundaries of that region. The boundary difference equations will generally have a different structure and hence, different stability properties, than the interior difference equations. There is no reason to believe that the boundary difference equations are less susceptible to a local Fourier analysis than the interior difference equations; in either case, only a given mesh point and its immediate neighbors are involved.

The preceding example involves only a single equation. Clearly, however, the method can be applied equally well to a system of equations. Again, three separate stability analyses are required: one for the equation system at a representative interior point and one for each boundary point. Each of these three analyses will involve an amplification matrix [2] rather than a scalar amplification factor. The stability condition is that no eigenvalue of any of these three matrices have a modulus exceeding unity.

The method presented here must be described as heuristic in that no rigorous conditions for its validity have been established. However, the method has been applied successfully to a number of special cases. Moreover, we reemphasize that the conventional Fourier method is usually applied in practice as a local method and hence, is equally heuristic. Therefore, we conjecture that the validity of the Fourier method, when applied in a linearized local manner, is essentially the same for boundary points as for interior points.

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